THE
MATHEMATICAL GAZETTE.

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NEWTON.

ISAAC NEWTON's investigations will always rank among the chief achievements of our race. It is intended in this sketch to give an outline of his life and what he did.

Newton was born prematurely on the morning of Christmas Day, 1642, in the manor house of Woolsthorpe, a tiny hamlet a few miles south of Grantham. His mother had been married less than a year, and was already a widow; her means were narrow, but when he was three years old she espoused a neighbouring parson, it being apparently a condition of the contract that Isaac should not accompany her to her new home, but should be given by the bridegroom a field to be added to the Woolsthorpe estate. His grandmother, Mrs. Ayscough, then came to live at the manor, and took charge of him: he continued with her till he was twelve years old, attending the village schools, where he learnt reading, writing, and arithmetic.

In 1655 he was sent to the Grammar School at Grantham, and boarded with a chemist in the town, so that he might get his education free. Here he seems to have come under an excellent master, one Henry Stokes. Newton was described as a "sober, silent, thinking lad," respected but not popular. What most impressed his school-fellows was his skill in drawing and in constructing toys, clocks, kites, etc. He already showed scientific aptitude rare in a boy of his age: for instance, in making his kites he tried all kinds of shapes, and investigated by experiment the best point to which to attach the string—these kites acquired local fame, for at night he flew them with lanterns attached, which frightened some of the country folk.

In 1656 his stepfather died, and his mother, returning to Woolsthorpe, brought him home, intending to bring him up as a farmer. For this he was quite unfitted, and she, realising the fact, sent him back to his former school. There he stayed till 1661, pursuing the normal course, including Latin and perhaps some logic, but apparently no mathematics beyond the elements of arithmetic. He continued to make experiments on various questions likely to excite the interest of an intelligent lad. For instance, he calculated the velocity of the wind in the great gale of 3 September, 1658, by measuring the length of the jumps he could make with and against the wind, and comparing them with the distance he could jump on a calm day: he also made more than one sun-dial. At school he was regarded as an exceptional genius, but such reputations are not worth much until tested in a wider world.

In 1661 he entered Trinity College, Cambridge, commencing residence in the summer, when he was about $18\frac{1}{2}$ years old. He matriculated as a sizar, for his means were small, the manor being worth only some £30 a year, but his mother and uncle seem to have been in a position to give him all that was necessary to enable him to prosecute his studies. I think of him at this time as a reserved and rather awkward youth, his head teeming with ideas and notions, specially interested in religious, mechanical, chemical, and physical questions, but as yet, so far as we know, with no definite scheme of work or career before him. A University course is invaluable for stimulating intellectual development. The best instruction, wide freedom restricted only by friendly regulations, and untrammelled intercourse with the ablest contemporaries of similar age and temperament drawn from all parts of the country—these are the most important features. Of the first two Newton took full advantage, and, notwithstanding his reserve, he made some friends.

There is nothing in his note-books to suggest that he was occupied with mathematics or physics during his first year of residence. His study of these subjects seems to have originated from a visit he paid in October, 1662 (or possibly 1663), to Stourbridge Fair, at that time one of the most important marts in England, and held on the outskirts of Cambridge. There he happened to pick up a book on astrology, but could not understand it on account of the geometry and trigonometry introduced. This led him to buy a *Euclid*, which he called "a trifling book," as the propositions seemed to him obvious, and Oughtred's *Clavis*, a good text-book on the algebra and arithmetic then current, which "he did not entirely understand." He was, however, interested, and decided to take up the systematic study of mathematics. He seems to have begun with Descartes's *Geometry*, which proved, as one might expect, beyond his powers, a fact which piqued and stimulated him. In College he attended a class by his tutor, B. Pulleyne, on Kepler's *Optics*, and is said to have surprised the lecturer by mastering the book before the instruction on it began. He also attended the lectures of I. Barrow, and subsequently, on Barrow's advice, studied Euclid's *Elements* carefully, from which he stated he derived great benefit. Meanwhile he seems to have constantly pegged away by himself at Descartes's *Geometry*, and in time, though only after repeated failures, he worked through it. Later he read it and Oughtred's *Clavis* again. He also studied the works of Vieta, Schooten, and Wallis. It is reasonable to suppose that he could command the assistance of Pulleyne and Barrow when he wanted help, but most of his reading was done by himself. It would seem that during his undergraduate days he made himself familiar with Euclidean Geometry, Geometrical Conics, Algebra, Trigonometry, Analytical Geometry, and Analysis as then studied. He also occupied his leisure by self-devised experiments in optics and probably in chemistry.

At Cambridge, and later, he kept diaries or commonplace books, some of which have been preserved. In one of these, dated January, 1664 N.S., mixed up with extracts and analyses of propositions from Vieta, Schooten, and Wallis, there are notes on angular sections, the squaring of curves, crooked lines that may be squared; on the extraction of roots, particularly those in affected powers; calculations about musical notes; and observations on refraction, the grinding of spherical optic glasses, and the methods of rectifying the errors of lenses. In the Lent Term, 1664, he investigated the theory of lunar halos, a difficult subject for one who was largely self-taught. In another place Newton wrote: "By consulting an account of my expenses at Cambridge . . . I find that in the year 1664, a little before Xmas, I, being then Senior Sophister, bought Schooten's *Miscellanies* and Cartes's *Geometry* (having read this

geometry and Oughtred's *Clavis* clean over half a year before) and borrowed Wallis's works. . . . At such time I found the method of Infinite Series": this must have been shortly before he graduated.

Newton took his B.A. degree in the Lent Term, 1665 N.S. In that spring the plague appeared, and for a couple of years he lived mostly at home, though with occasional residence at Cambridge. Probably at this time his creative powers were at their highest. His use of fluxions may be traced back to May, 1665; his theory of gravitation originated in 1666; and the foundation of his optical discoveries would seem to be only a little later. In an unpublished memorandum made some years later (cancelled, but believed to be correct in the part here quoted), he thus described his work of this time: "In the beginning of the year 1665 I found the method of approximating Series and the Rule for reducing any dignity of any Binomial into such a series. The same year in May I found the method of tangents of Gregory and Slusius, and in November had the direct method of Fluxions, and the next year in January had the Theory of Colours, and in May following I had entrance into the inverse method of Fluxions. And the same year I began to think of gravity extending to the orb of the Moon, and . . . from Kepler's Rule of the periodical times of the Planets being in a sesquialterate proportion of their distances from the centers of their orbs I deduced that the forces which keep the Planets in their orbs must [be] reciprocally as the squares of their distances from the centers about which they revolve: and thereby compared the force requisite to keep the Moon in her orb with the force of gravity at the surface of the earth, and found them answer pretty nearly. All this was in the two plague years of 1665 and 1666, for in those days I was in the prime of my age for invention, and minded Mathematicks and Philosophy more than at any time since."

In 1667 Newton was elected to a fellowship, which secured him a bare competency, and in 1668 he definitely took up residence in Cambridge. There had been a boy and girl love affair between himself and a Miss Storey of Grantham, but nothing came of it, for he was a poor man mainly dependent on his fellowship, and the retention of that was conditional on celibacy. His resumption of residence in Cambridge opened a new epoch in his life. We may picture him at this time as a short, well-set man, with a broad forehead, a determined square jaw, bright blue eyes, and sharp features, with a longish nose. His hair, which had been dark brown, was already turning grey, but remained thick. Of course, like all men of his time, he shaved clean. His dress was slovenly. He was simple, generous, modest, scrupulously just, but reserved, self-absorbed, tenacious of his rights, and easily upset by any controversy.

The next period in Newton's career covers some thirty years, during which he lived in College, engrossed for the most part in researches which are briefly summarised below. Soon after his return to Cambridge he was elected to the Lucasian Chair of Mathematics. As professor it was his practice to lecture publicly once a week for from half an hour to an hour at a time in one term of each year, probably dictating his lectures as rapidly as they could be taken down; and in the week following the lecture to devote two or three hours to appointments which he gave to students who wished to come to his rooms to discuss the subject of the previous lecture. He never repeated a course, which usually consisted of nine or ten lectures, and generally the lectures of one course began from the point at which the preceding course had ended. The manuscripts of his lectures for seventeen out of the first eighteen years of his tenure are extant. His professorial lectures from 1669 to 1672 were on optics, from 1673 to 1683 on algebra, and from 1684

to 1691 on gravitation and allied subjects. Probably he supplemented his lectures by lending manuscripts to his pupils. The years 1684 to 1686 are memorable for his production of the *Principia*, but I defer for the moment any further description of his scientific work. During most of this time he also occupied himself with chemical experiments—ingenious though not fruitful—and devoted considerable time to theology, but we may regard these investigations as the relaxations of an able student, and his conclusions thereon need not detain us. As early as 1675 the value of his researches were widely recognised, and in that year the Crown issued letters-patent permitting him as a professor to continue to hold his fellowship without the necessity of taking orders, to which he felt scruples.

Throughout this period his annoyance at interruptions seems to have steadily increased, until it became an obsession which his friends tried in vain to overcome. No doubt it had been stimulated by the discourtesy or worse of Hooke and Linus between 1672 and 1675 and of Hooke in 1684. An amanuensis, whom Newton employed from 1683 to 1689, describes him in the latter year as "very meek, sedate, and humble, never seemingly angry, of profound thought, his countenance mild, pleasant, and comely"; also as never laughing, always keeping close to his studies, never taking any recreation or exercise, so intent on his work that he ate but little, and rarely entertained or allowed himself to be entertained. This somewhat unattractive portrait represents him as seen by his friends, and its general accuracy is confirmed from other sources. To this description I may add, that though he was absent-minded, yet when allowed to do things in his own way his shrewdness and capacity were generally recognised. In 1673 and again in 1686 he took a prominent part in academic affairs, and later represented the University in Parliament for a short time. In 1692 he suffered from insomnia, caused by overwork. Gradually, however, rest, his satisfaction at the acknowledged value of his discoveries, and the kindly affection of his friends had their effect in completely restoring him to health. In 1693 he commenced investigations designed to extend and complete the lunar theory as far as might be, but this was destined to remain unfinished.

In 1696 he left Cambridge, and the rest of his career covers some thirty years during which he lived in London, holding in succession the offices of Warden and Master of the Mint. He was now comparatively wealthy, which gave him opportunities he much valued for generosity, enjoyed a well-appointed home, knew everyone whom he desired, and was universally honoured and esteemed, while, being past middle life, his fits of abstraction and his untidiness in dress and manners were looked on as mere eccentricities of genius. In such circumstances his character mellowed.

At this time he became involved in two controversies: one, lasting from 1705 to 1724, concerned with whether Leibnitz had discovered the infinitesimal calculus independently or had appropriated the idea from Newton; the other, lasting from 1705 to 1712, about the publication of Flamsteed's Greenwich observations. In both disputes the general opinion of contemporaries was in favour of Newton. Save for the annoyance caused by these discussions, the latter years of his life were very happy. Reports that he wrote on official matters and on questions referred to him show him as an acute and well-informed man of the world. In London he occupied much of his leisure with the question of the Old Testament prophecies, but in science he produced nothing more of special note.

Several of his earlier investigations were now published. Previously, save for some early memoirs on light, 1672 to 1675, the *De Motu*, 1684,

and the *Principia*, 1686, he had published little, mainly, it would seem, owing to his fear of being forced into controversies. Now, however, his friends induced him, albeit somewhat reluctantly, to allow the publication in 1700 of his *Quadrature of Curves*, in 1702 of his *Lunar Theory*, in 1704 of his *Optics*, with appendices on Cubics and the Quadrature of Curves, in 1707 of his *Universal Arithmetic*, in 1711 of his *Analysis* by equations with an infinite number of terms, written in 1669, and his *Differential Method* on interpolations, in 1712 of his *Analysis* by series, fluxions, and differences; and in 1713 of a second and enlarged edition of his *Principia*. A third edition of the *Principia* was issued in 1726; his *Optical Lectures*, delivered in 1669, were published in 1728, and those delivered in 1670 and 1671 in 1729, and his *Fluxional Calculus*, unfinished but mostly written in 1670, was published by Colson in 1736.

Except that his hair became white and that he grew broader and stouter, his appearance changed but little as age advanced. He died in his 85th year on 20 March, 1727 N.S., was honoured by a state funeral, and buried in Westminster Abbey. He had been knighted in 1705.

I now proceed to describe successively the chief results of his work in Pure Mathematics, in Optics, in Mechanics and Gravitation, and in Motion in a Resisting Medium. This classification is not exhaustive, but it will serve my purpose, which is rather to indicate his achievements than to describe them. The order of his discoveries, where they were first published, and details about them will be found in histories of mathematics.

It is probable that Newton was interested in Pure Mathematics mainly as an instrument of research, but none the less it will be useful briefly to mention his discoveries in Euclidean Geometry, Algebra, Expansions, Analytical Geometry, Prime and Ultimate Ratios, and Fluxions. In general his results or demonstrations were given in their finished form, and we have no clue as to their origin or the processes used originally in obtaining them.

His command of the processes of Pure Geometry has rarely been equalled. In the *Principia* he gave his demonstrations in the language of Euclidean Geometry. Probably this was wise, for the principles of the calculus had not then been expounded, and had he used it, a discussion as to the truth of his conclusions would have been hampered by a dispute concerning the validity of the methods employed in proving them. The translation of results obtained by the use of fluxions and analysis into a geometrical form was an amazing feat, involving the establishment of numerous theorems of great difficulty, but no one to-day would consider this the best way of presenting the subject.

His work on Algebra and the Theory of Equations is more familiar. In his celebrated letter to Leibnitz in 1676 he enunciated the binomial theorem for a fractional index, obtaining the result in the first instance by the method of interpolation. He verified his conclusion by testing it in various ways: for instance, in the case of $(1-x^2)^{\frac{1}{2}}$ by first extracting the square root of $1-x^2$ by the usual algebraic process and seeing whether it agreed with his formula, and next by squaring his result and seeing if it reduced to $1-x^2$. In his lectures he introduced the use of literal indices, and established a considerable number of propositions dealing with roots of algebraical equations, gave geometrical illustrations, and showed that complex roots occur in pairs. His theorem for finding the sum of the nth powers of the roots is well known, and laid the foundation of the theory of symmetrical functions of the roots of an equation. He extended Descartes's rule of signs to give limits to the number of complex roots.

Newton was much interested in methods of obtaining an Expansion of

a Function of x in an infinite series in ascending powers of x. The binomial theorem was one instance of this, obtained by the method of interpolation. Another is found in his expansion of $\sin^{-1}x$. From the latter result he deduced by reversion of series the expansion of $\sin x$: this is the earliest known use of the method of reversion of series. To-day these methods have been largely superseded by the use of Taylor's Theorem. Early in his career he used these expansions to express the ordinate of a curve in terms of the abscissa in an infinite series, and thus obtained expressions in infinite series for the rectification of arcs, the quadrature of curves, etc.

His tract on *Cubic Curves*, of which the substance was perhaps written in 1669, is interesting as showing that he was also proficient in analytical geometry. Its application to conics was already familiar to expert mathematicians: in applying it to cubics he broke new ground. He began by classifying curves into algebraical and transcendental, and discussed the theory of asymptotes, curvilinear diameters, double points in the plane and at infinity, and the graphical solution of problems by the use of curves. Applying his results to cubics, he divided them into four classes, which contained in all 72 species: these he discussed in detail—there are, in fact, 78 species according to this system of classification. He added the remarkable theorem that just as all conics can be obtained from the shadow of a circle cast on a shifting plane by a luminous point, so all cubics can be obtained from the shadow of a certain parabolic cubic: this remained an unsolved puzzle until 1731.

We may also include among his work in pure mathematics his enunciation of the Principle of Prime and Ultimate Ratios as set out in the *Principia*. This enabled him to integrate certain functions by the use of geometry, and thus avoid the difficulties of the method of indivisibles as employed by Cavalieri and others. This has been rendered obsolete by the acceptance of the principles of the calculus.

The introduction of the Infinitesimal Calculus was one of the great intellectual achievements of the seventeenth century. The calculus was invented by Newton, but there are isolated cases of its use in particular theorems by previous writers. The general method, expressed in the notation of fluxions and fluents, was used by Newton as early as 1665 and 1666, and was communicated to friends and pupils, but his complete exposition was published only in 1736, though references to or descriptions of it appeared in 1686 in the *Principia* (book ii. lemma ii.), in 1693 in the second volume of Wallis's *Works*, in 1700 and 1704 in the *De Quadratura*, and in 1712 in his *Analysis* by series, fluxions, and differences.

The idea of a fluxion or differential coefficient, as treated at this time is simple. When two quantities—for instance, the radius of a sphere and its volume— are so related that a change in one causes a change in the other, the one is said to be a function of the other. The ratio of the rates at which they change is termed the differential coefficient or fluxion of the one with regard to the other, and the process by which this ratio is determined is known as differentiation. Knowing the differential coefficient and one set of corresponding values of the two quantities, it is possible by summation to determine the relation between them, but often the process is difficult. If, however, we can reverse the process of differentiation, we can directly obtain this result, which is known as the integral or fluent: the process of reversal is termed integration. It was at once seen that problems connected with the rectification and quadrature of curves and the determination of volumes were reducible to integration. In mechanics also, by integration velocities could be deduced from known accelerations, and distances traversed from known

velocities. In short, wherever things change according to known laws, here was a possible method of finding the relation between them. It is true that when we try to express observed phenomena in the language of the calculus we usually obtain an equation involving the variables and their differential coefficients, and possibly the solution may be beyond our powers, but even so, the method is often fruitful.

The modern presentation of the process of differentiation rests on finding the ratio of two quantities, both of which ultimately vanish. This involves some awkward questions of philosophy which before Weierstrass's researches were usually slurred over. Newton, however, evaded this difficulty by assuming that all magnitudes might be conceived as generated by motion, for instance, a solid by the motion of a surface, a plane angle by the rotation of a line, that is, he regarded everything as a function of time whose passing is uniform. If the motion be continuous, so is the function. But this only postponed the difficulty, for in applying his results Newton was forced to introduce infinitesimals or, as he called them, "moments of the fluent."

I turn next to the subject of Newton's work in Applied Mathematics and Physics. Here I shall mention in succession his researches in Optics, Mechanics, and Gravitation, and Motion in a Resisting Medium. His conclusions were based on the sure foundation of facts obtained by experiment, and not on metaphysical conjectures or *à priori* assumptions. The skill he showed in devising experiments, the accuracy of his observations, and his caution in drawing inferences are remarkable: for his astronomical data he was indebted to practical astronomers.

The subject of Optics was specially attractive to Newton. Even before his appointment to the Lucasian Chair he had discovered how to decompose, by means of a glass prism, a ray of solar light into rays of different colours; in other words, he showed that the index of refraction of a substance varied with the colour of the light used. Thus colour was for the first time brought within the domain of science. The complete explanation of the theory of the rainbow followed. In his early lectures he described these experiments, as well as those for determining the index of refraction for a given substance for rays of particular colours. He also simplified and extended the geometrical theory of reflexion and refraction at curved surfaces. By a curious chapter of accidents he failed to correct the chromatic aberration of two colours by means of a couple of prisms. He therefore abandoned the hope of making a refracting telescope which should be achromatic, and instead designed in 1668 a reflecting telescope: the form he used is that still known by his name; in 1672 he invented a reflecting microscope, and somewhat later a sextant in the form rediscovered by J. Hadley in 1731.

His investigations in geometrical optics led him to consider how the effects of light were produced. Two hypotheses naturally suggest themselves—one that the source of light moves or affects the medium (or ether) between it and the eye by pulsations or waves; the other that the source of light throws out something (maybe streams of corpuscles) whose impact on the eye gives the sensation of light. The former or undulatory or wave theory appears the more natural, and enables us at once to account for reflexion, refraction, and many of the more obvious phenomena. In Newton's time, however, it was not seen how the rectilinear propagation of light and the phenomena of polarisation could be brought under it: nor was this done until about a century ago, when physicists suggested that the vibrations in the ether were entirely transverse to the direction of the ray, and expounded the principle of interference. The second or corpuscular theory enables us to explain the rectilinear propagation of light, reflexion, refraction, and allied problems. It is not

free from objections, but it did account for the facts then known, and at the time presented less difficulty than the wave theory.

Newton definitely rejected the wave theory, but he never fully accepted the corpuscular theory which is commonly associated with his name. In fact, the assumption of the existence of material corpuscles, for which no other evidence exists, expelled in immense numbers by a source of light, was repugnant to his idea of legitimate scientific conjecture, though he regarded it as a possible scheme. As his views have been often misunderstood, I describe them at length. His opinion was that all space is permeated by an elastic ether capable of transmitting vibrations, that this ether pervades all bodies and is not necessarily uniform, that possibly electricity and gravitation may be due to it, though in what way we do not know, that it may be essential to the production of light, but that light cannot be due to its vibrations since light rays travel in straight lines. Light, he went on to say, is "something propagated by lucid bodies." It may be "an aggregate of peripatetic qualities," or it may arise from "multitudes of unimaginable small and swift corpuscles," or it may be "any other corporeal emanation or any impulse or motion of any other medium . . . diffused through the main body of ether," or anything else which they that like not these views "can imagine proper for the purpose. . . . Let every man here take his fancy: only whatever light be, I suppose it consists of rays differing from one another in contingent circumstances, as bigness, form, or vigour." Of these vague hypotheses, that referring to corpuscles was the simplest: it was generally adopted by Newton's followers, and commonly attributed to him, though, in fact, his object seems to have been to present a theory free from speculation as to the mechanism that produced the phenomena. In the *Principia* Newton went somewhat further, and said that whatever light is, there is at any rate a close analogy between rays of light and streams of corpuscles, but he does not seem to have ever committed himself to more than saying that the existence of corpuscles was one possible way of explaining the phenomena. Subject to these remarks, we may for brevity express the theory as though it assumed the existence of corpuscles, and in this sense may say that Newton regarded colour as an inherent characteristic of the corpuscle.

An initial difficulty occurs in the theory owing to the fact that when a ray of light strikes (say) a sheet of water some of the light may be reflected and some retracted. To account for this Newton was forced to modify the simple corpuscular hypothesis by the assumption that the corpuscles pass periodically through states of easy reflexion and easy refraction. This conjecture, though ingenious, was artificial, and later further assumptions had to be introduced in order to enable other phenomena to be explained. Any hypothesis which has to be supported by constant complicated additions must be regarded with suspicion. In this case a crucial test existed, for according to the corpuscular theory the velocity of light is greater in denser mediums, while according to the undulatory theory it is less. It was not until the nineteenth century that physicists were able to devise experiments to determine the velocity of light in water as compared with the velocity in air. When they did so they found that the result agreed with the undulatory and not with the corpuscular theory: thus the experiment finally disposed of the latter theory. Most of Newton's optical researches were embodied in papers communicated to the Royal Society between 1672 and 1675.

I turn next to describe briefly Newton's work on Mechanics and Gravitation as expounded in the *Principia*. The work is a classic in the history of mathematics, and its publication profoundly affected the

methods of scientific research and the ideas of men about the universe. It commences with definitions of mass, momentum, force, and acceleration: these are followed by the enunciation of the three laws of motion. Of these the first two, relating to the motion of a particle, had been, in effect, given by Galileo, but the third, which enabled the science to be applied to the motion of collections of particles or bodies of definite size, was new. Newton's account of his experiments and his exposition of the principles are concise, but a large part of the work may be considered as the application of these laws.

The theory of gravitation as here set out arose from an investigation of the cause of the motion of the planets round the sun. The origin of this theory has been often told, but will bear repetition. The fundamental idea occurred to Newton in 1666. His reasoning was as follows. He knew that gravity extended to the highest hills, he saw no reason why it should cease to act at greater heights, accordingly he believed that it would be found in operation as far as the moon, and he suspected that it might be the force which retained that body in its path round the earth. This opinion he verified thus. If a stone is allowed to fall near the surface of the earth, the attraction of the earth causes it to move through sixteen feet in one second. Also Kepler's Laws, if accurate and applicable, involve the conclusion that the attraction of the earth on a distant body varies inversely as the square of its distance from the centre of the earth. He knew the radius of the earth and the distance of the moon, and therefore, on this hypothesis, could find the magnitude of the earth's attraction on the moon. Further, assuming that the moon moved in a circle, he could calculate the force required to retain it in its orbit. The results agreed "pretty nearly." The discrepancy was caused by the fact that at this time his estimate of the radius of the earth was inaccurate, and that thus the force which he calculated to be necessary to retain the moon in its orbit was rather greater than the earth's attraction. But his faith in his theory was unshaken, though he conjectured that the moon's motion was also affected by other influences, such, for example, as Descartes's hypothetical vortices.

In 1679, in consequence of some correspondence about the path of a falling body, Newton was led to take up the subject again. By this time he knew with approximate correctness the length of the radius of the earth, and, on repeating his calculations, he found the results to be in accord with his former hypothesis. He then proceeded to the general theory of the motion of a particle under a force directed to a fixed point, and showed that the vector to the particle would sweep over equal areas in equal times. He also proved that if a particle describes an ellipse under a force directed to a focus, the law must be that of the inverse square of the distance from the focus, and conversely that the orbit of a particle projected in free space under the influence of such a force must be a conic. The application to the solar system was obvious, since Kepler had shown that the planets describe ellipses with the sun in one focus, and that the vectors from the sun to them sweep over equal areas in equal times. This investigation was made for his own satisfaction, and was not published at the time. In it he treated the moon as if it were a particle, and must have realised that the results could be taken as being only approximately correct.

In 1684 the subject of the planetary orbits was discussed in London by Halley, Hooke, and Wren. They were aware that, if Kepler's conclusions were correct, the attraction of the sun or earth on a distant external particle must vary inversely as the square of the distance, but they could not determine the orbit of a particle subjected to the action

of a central force of this kind. It occurred to Halley and Wren that Newton might be able to assist them. Accordingly, in August, Halley went to Cambridge for a talk on the subject, and found that Newton had solved the problem some five years previously. At Halley's request, Newton wrote out the substance of his argument, showing that the path was a conic, and sent it to London. Halley recognised its significance, and later in the autumn returned to Cambridge, and persuaded Newton to promise to attack the general problem and to allow his results to be published.

As yet Newton had dealt with the problem as if the sun and the planets might be regarded as heavy masses concentrated at their centres. Clearly at the best this was only an approximation, though, considering the enormous distances involved, it was not unreasonable. In January or February, 1685, he considered the question of the attraction of bodies of finite size, and was delighted to find that a uniform sphere or spherical shell attracts an external particle as if condensed into a heavy mass at its centre. Hence the results he had already proved for the relative motion of particles were true for the solar system, save for small errors, due partly to the fact that the bodies were not perfectly spherical, and partly to disturbances caused by the planets attracting one another. It was no longer a question of rough approximation; the problem was reducible to mathematical analysis, subject to the introduction of relatively minute corrections, which, given the necessary observations, could be calculated very closely.

The first book of the *Principia* deals with the motion of particles or bodies in free space, either in known orbits or under the action of known forces. In it the law of attraction is generalised into the statement that every particle of matter attracts every other particle with a force which varies directly as the product of their masses and inversely as the square of the distance between them. Thus gravitation was brought into the domain of science, but what caused it Newton did not profess to know, and here, as in his theory of light, it was his object to present the theory free from speculation as to the mechanism that produced the phenomena.

Having investigated the general theory, Newton later applied the results to the chief phenomena of the solar system, and he showed that the facts then known about it and in particular the path of the moon, with its various inequalities, the figure of the earth, and the motion of the tides, were in accord with the theory. Much of the material for these calculations was collected for him by Flamsteed and Halley.

Newton prefaced these applications of the theory with four rules which should guide scientific men in making hypotheses. These, in their final shape, are to the following effect: (1) We should not assume more causes than are sufficient and necessary for the explanation of observed facts. (2) Hence, as far as possible, similar effects must be assigned to the same cause; for instance, the fall of stones in Europe and America. (3) Properties common to all bodies within reach of our experiments are to be assumed as pertaining to all bodies; for instance, extension. (4) Propositions in science obtained by wide induction are to be regarded as exactly or approximately true, until phenomena or experiments show that they may be corrected or are liable to exceptions. The substance of these rules is now accepted as the basis of scientific investigation. Their formal enunciation here serves as a landmark in the history of thought.

Newton's conclusions were immediately accepted in Great Britain, and rapidly by leading mathematicians in Europe. The refutation of the Cartesian hypothesis, ran, however, counter to the sentiments and wishes of certain writers, and some few years elapsed before the truth of the

gravitation theory was universally admitted, but it would be ungracious to dwell on this. Broadly speaking, we may say that the argument was almost at once recognised as conclusive. In Great Britain the work exercised a profound influence in philosophy as well as in science, and educated men of all schools of thought acquainted themselves with the general line of Newton's reasoning. The influence of the *Principia* on dynamical astronomy has been permanent. Perhaps it is not too much to say that when it was published, the theory, as there set out, had outstripped observation, but during the succeeding century large numbers of new facts were collected, and applications of the theory to new problems were made, notably by Clairaut, Euler, Lagrange, and Laplace. All these researches tended to confirm the theory. It is true that Newton applied his theory only to the solar system, for which alone he had the necessary data, but after the publication of the *Principia*, no one doubted that gravity extended to the most distant regions of space. The work of Sir William Herschel and that of all later astronomers on binary and other systems rests on this hypothesis.

On the value of the *Principia* it is interesting to quote remarks by two of the most eminent mathematicians of subsequent times. Lagrange described it as the finest production of the human mind, and added that Newton was not only the greatest genius that had ever existed, but also the most fortunate, for as there is but one universe, it can happen but to one man in the world's history to be the interpreter of its laws: while Laplace deliberately expressed the opinion that the *Principia* was pre-eminent over every extant production of human genius—"so near the gods, man cannot nearer go."

A few words must be added on Newton's work on the Theory of Motion in a Resisting Medium—the science of hydrodynamics—involving the discussion of wave-motion, the theory of sound or waves in the air, the undulatory theory or waves in the ether, and vortex motion. This is set out in the second book of the *Principia*. It has hardly received due acknowledgment, in fact, at the time it was overshadowed by the theory of gravitation and the applications to the solar system. The theory is difficult, and when given in the seventeenth century was necessarily incomplete. That Newton could carry his calculations so far with the limited mathematics then at his command is not the least wonderful part of the performance, but it is the prerogative of genius to get results with scanty equipment.

In the course of the work Newton raised the question of the form of the solid of revolution which would encounter the least resistance in moving through a medium whose resistance is proportional to the square of the velocity, a usual assumption which in many cases is approximately true. He gave a construction equivalent to saying that if the axis of x be taken as the direction of motion and the axis of revolution, then the differential equation of the generating curve is $y=c(1+\dot{y}^2)^2/\dot{y}^3$. This result baffled his readers, and it was not until years had elapsed that a demonstration of it was discovered. J. C. Adams, however, in going through the Portsmouth Papers a few years ago, found a copy by Newton of a letter he had written to David Gregory in or about 1694 explaining how he had arrived at this result. Newton used fluxions in obtaining it, and as he made it a rule to give in the *Principia* none but geometrical demonstrations, it may be conjectured that it was the difficulty of presenting the proof in a geometrical form that led him to give only the result.

This bare description of Newton's chief researches will explain and justify the great influence he exercised on scientific thought and methods in his own time and for many years after his death. Of his unrivalled

intellectual powers there is no question. He was deeply religious—"the whitest soul I ever knew," said Bishop Burnet—strictly just to himself as to others, candid, and affable, but unforgiving anything he deemed unfair. His modesty is well exemplified by his remark, made just before his death, that to himself he seemed to have been only like a boy playing on the seashore, and diverting himself in now and then finding a smoother pebble or prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before him. So Newton stands, one of the great figures in the history of science. W. W. ROUSE BALL.

MATHEMATICAL NOTES.

419. [C. 1. g.] *A Proof of Vandermonde's Theorem and a Generalisation.*

Denote $x(x-1)(x-2)\ldots(x-n+1)$ by $x^{(n)}$ and assume, as is obviously possible, that

$$\frac{(x+y)^{(n)}}{n!}=a_0+a_1x+\frac{a_2x^{(2)}}{2!}+\ldots+\frac{a_nx^{(n)}}{n!}, \qquad \ldots\ldots(1)$$

where $a_0, a_1, \ldots a_n$ are functions of y only.

Differencing equation (1) r times with respect to x, we obtain

$$\Delta^r\frac{(x+y)^{(n)}}{n!}=\frac{(x+y)^{(n-r)}}{(n-r)!}=a_r+a_{r+1}\frac{x}{1!}+\ldots a_n\frac{x^{(n-r)}}{(n-r)!}. \qquad \ldots\ldots(2)$$

By putting $x=0$, a_r is at once seen to be $\dfrac{y^{(n-r)}}{(n-r)!}$.

Thus, $$\frac{(x+y)^{(n)}}{n!}=\frac{y^{(n)}}{n!}+\frac{y^{(n-1)}\cdot x}{(n-1)!\,1!}+\frac{y^{(n-2)}x^{(2)}}{(n-2)!\,2!}+\ldots+\frac{x^{(n)}}{n!}.$$

Similarly, by assuming

$$\frac{(x+y)^{(n)}}{n!}=a_0+a_1(x+\alpha_1)+\frac{a_2(x+\alpha_2)^{(2)}}{2!}+\ldots+\frac{a_n(x+\alpha_n)^{(n)}}{n!},$$

where α_1 is perfectly arbitrary, $\alpha_2-\alpha_1=0$ or 1, $\alpha_3-\alpha_2=0$ or 1, and generally $\alpha_{r+1}-\alpha_r=0$ or 1 for all values of r from 1 to $n-1$, we can prove that

$$\frac{(x+y)^{(n)}}{n!}=\frac{(y-\alpha_1)^{(n)}}{n!}+\frac{(y-\alpha_2)^{(n-1)}(x+\alpha_1)}{(n-1)!\,1!}+\frac{(y-\alpha_3)^{(n-2)}(x+\alpha_2)^{(2)}}{(n-2)!\,2!}+\ldots+\frac{(x+\alpha_n)^{(n)}}{n!}.$$

S. T. SHOVELTON.

420. [L¹. 6. a.] *To find the centre of curvature of the conic $ax^2+by^2=1$ at the point (x', y').*

If the normals at the extremities of the two chords $lx+my-1=0$ and $l'x+m'y-1=0$ meet in a point (h, k), then $\dfrac{ll'}{a}=\dfrac{mm'}{b}=-1$.

Also, if we take $lx+my-1=0$ as the tangent at (x', y'), the point (h, k) will be the centre of curvature required. Hence

$$ax^2+by^2-1=(axx'+byy'-1)\left(\frac{x}{x'}+\frac{y}{y'}+1\right)$$

must be the same as $xy(a-b)-akx+bhy=0$.

The above equation becomes

$$xy\left(\frac{ax'}{y'}+\frac{by'}{x'}\right)+x\left(ax'-\frac{1}{x'}\right)+y\left(by'-\frac{1}{y'}\right)=0$$

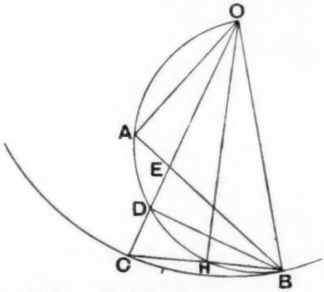
or $$xy-bxy'^3-ayx'^3=0,$$

whence $$h=-\frac{a(a-b)}{b}x'^3, \quad k=\frac{b(a-b)}{a}y'^3.$$

N. M. GIBBINS.

421. [K. 21. b.] (v. *Note* 393, *p.* 108.) Let BC meet semi-circle at H. Since $ED=DC$ and the angle EDB is a right angle,

$$\therefore\ DB \text{ bisects } \angle ABC;$$
$$\therefore\ \text{arc } AD = \text{arc } DH.$$



Since $OC=OB$ and OHB is a right angle,

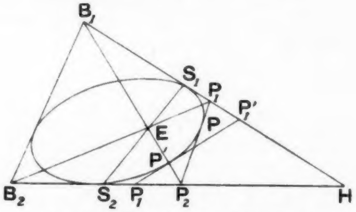
$$\therefore\ OH \text{ bisects the } \angle COB;$$
$$\therefore\ \text{arc } DH = \text{arc } HB;$$

that is, OD, OH trisect the angle AOB.

(Solution by several boys at Bancroft's School, Woodford Green.)

422. [L¹. 2. a.] Let four tangents be drawn to a conic at S_1S_2BP.

Then $\left.\begin{matrix}B_2P_1\\B_1P_2\end{matrix}\right\}$, *i.e.* E lies on polar of H, *i.e.* on S_1S_2;



$$\begin{aligned}\therefore\ B_2\{P_1P_1'P_1''\dots\} &= B_2\{EE'E''\dots\}\\ &= B_1\{EE'E''\dots\}\\ &= B_1\{P_2P_2'P_2''\dots\},\end{aligned}$$

i.e. ranges $\{P_1P_1'P_1''\dots\}$, $\{P_2P_2'P_2''\dots\}$ are homographic. S. ANDRADE.

423. [I. 18.] *Formulae for Three Cubes whose Sum is a Cube.*

(i) $$(2x^2-4xy+42y^2)^3+(x^2+16xy-21y^2)^3+(21y^2+16xy-x^2)^3 = (2x^2+4xy+42y^2)^3.$$

E.g., if $x=3$ and $y=1$, we have $48^3+36^3+60^3=72^3$, or $4^3+3^3+5^3=6^3$.

If $x=10$ and $y=1$, $202^3+239^3+81^3=282^3$.

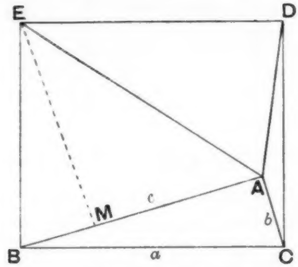
(ii) $$(x^2-7xy+63y^2)^3+(8x^2-20xy-42y^2)^3+(6x^2+20xy-56y^2)^3 = (9x^2-7xy+7y^2)^3.$$

If $y=0$, $1^3+8^3+6^3=9^3$. If $x=4$ and $y=1$, it reduces to

$$17^3+2^3+40^3=41^3.$$

G. OSBORN.

424. [K^1. 3. c.] Modification of the proof given by Mr. Hawkins, *Gazette*, vol. v. p. 143. Complete the square $BCDE$ on the hypotenuse, on the side towards A, and join AE, AD.



The perpendicular EM on AB is equal to AB, for the triangles BEM, CBA are congruous.

$\therefore$ area of triangle $ABE=\frac{1}{2}c^2$;

likewise " " $ACD=\frac{1}{2}b^2$.

The two triangles AED, ABC have equal bases each a, and the sum of their heights is a, so their combined area is $\frac{1}{2}a^2$.

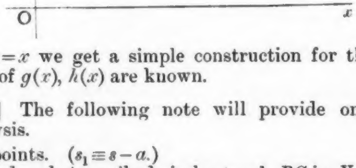
But the four triangles make up the square, so

$$a^2=\frac{a^2+b^2+c^2}{2} \quad \text{or} \quad a^2=b^2+c^2.$$

C. S. J.

425. [X. 4.] *Note on graphing.*

From any point F on $y=f(x)$ draw FG parallel to Oy to cut $y=g(x)$ in G and FH parallel to Ox to cut $y=h(x)$ in H. Complete the rectangle $FGPH$; then P is clearly a point on $y=gf^{-1}h(x)$.



By taking $f(x)=x$ we get a simple construction for the graph of $gh(x)$ when the graphs of $g(x)$, $h(x)$ are known. E. J. Nanson.

426. [K^1. 1. b.] The following note will provide one or two simple examples in analysis.

The (s_1, s_2, s_3) points. $(s_1 \equiv s-a.)$

Let the inscribed and A-escribed circles touch BC in X and X_1.

AX, BY, CZ are concurrent at a point M, called the Gergonne Point of ABC, with *barycentric* coordinates $1/s_1, \ldots$.

AX_1, BY_2, CZ_3 are concurrent at the Nagel Point, with barycentric coordinates $s_1, \ldots$.

The centre of similitude σ of the homothetic triangles XYZ, $I_1I_2I_3$ lies on OI: its *normal* coordinates being $1/s_1, \ldots$.

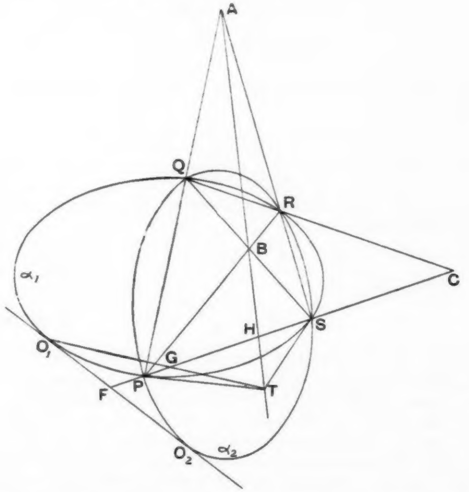
Denote by K_i the point whose *normal* coordinates are $s_1, \ldots$.

The perpendiculars from K_i on the sides of $I_1I_2I_3$ are as these sides, and therefore K_i is the Lemoine Point of $I_1I_2I_3$.

Prove that σ, M, G, K_i are collinear: also that MN cuts OI at the orthocentre of XYZ, and passes through the isotomics of I and I_1 with b, c, $1/a, \ldots$ and $\cot A, \ldots$.

W. GALLATLY.

427. [L^1. 2. a.] *Let O_1 and O_2 be two conjugate points for the pencil of conics through P, Q, R, S. Let PQ, RS meet in A, PR, SQ meet in B, and PS, QR in C. Then $O_1(PQRS) = O_2(O_1ABC)$.*



Of the pencil of conics, if we draw the two a_1, a_2 passing respectively through the points O_1, O_2, they will have O_1O_2 for a common tangent, and T, the pole of PS for a_1, lies on AB. Let PS meet O_1O_2 in F, O_1T in G, and AB in H. Then F is the pole of O_1T for a_1, and therefore F and G are conjugate points (1).

Then the conic-pencil $O_1(PQRS) = P(PQRS) = (TABH)$.

Again, the conic which is the locus of the poles of the line O_1O_2 for the pencil of conics passes through the points A, B, C, O_1, O_2, and also through the point G, since G is the fourth harmonic of F for S and P, by (1);

$$\therefore \text{ the conic-pencil } O_2(O_1ABC) = G(O_1ABC) = (TABH) = O_1(PQRS).$$

JOHN J. MILNE.

428. [K. 7. d; M^1. e. δ.] *Let O_1 and O_2 be two conjugate points with respect to the four-point system of conics passing through P, Q, R and S. Let A be the intersection of PQ and RS, B of PR and QS, C of PS and QR. To prove that $O_1[PQRS] = O_2[O_1ABC]$.*

This enunciation is quoted from Dr. Milne's note (*Gazette*, vol. v. p. 386), with a slight alteration. His method did not enable him to distinguish readily between the six anharmonic ratios of a pencil, and he wrote $O_2[ABCO_1]$ instead of $O_2[O_1ABC]$.

Let the homogeneous coordinates of P, Q, R, S be

$$(1, 1, 1), \quad (-1, 1, 1), \quad (1, -1, 1), \quad (1, 1, -1) \text{ respectively.}$$

Any conic of the system is $lx^2+my^2+nz^2=0$ with the condition $l+m+n=0$. Hence O_1 and O_2 are conjugate if their coordinates are related in the form

$$x_1x_2=y_1y_2=z_1z_2.$$

Now the points where the lines joining O_1 to P, Q, R, S cut $x=0$ are given by the four equations:

$$y(x_1\mp z_1)-z(x_1\mp y_1)=0,$$
$$y(x_1\mp z_1)+z(x_1\pm y_1)=0.$$

Hence $$O_1[PQRS]=\frac{x_1^2-z_1^2}{x_1^2-y_1^2}.$$

Again O_2O_1 cuts $x=0$ where $y(x_1z_2-x_2z_1)=z(x_1y_2-x_2y_1)$, and O_2A cuts it where $yz_2=zy_2$.

Hence $$O_2[O_1ABC]=\frac{x_1z_2-x_2z_1}{x_1y_2-x_2y_1}\cdot\frac{y_2}{z_2}=\frac{x_1^2-z_1^2}{x_1^2-y_1^2}$$
$$=O_1[PQRS].$$

It may be noticed that with this notation Dr. Milne's cubic and polar conic have the real equations:

$$\begin{vmatrix} x^2 & y^2 & z^2 \\ xx_1 & yy_1 & zz_1 \\ 1 & 1 & 1 \end{vmatrix}=0 \quad\text{and}\quad \begin{vmatrix} x^2 & y^2 & z^2 \\ x_1^2 & y_1^2 & z_1^2 \\ 1 & 1 & 1 \end{vmatrix}=0.$$

F. J. W. WHIPPLE.

429. [V. a. θ.] *The Teaching of Numerical Trigonometry.*

To the *Gazette* of December, 1913, which has just reached me, Mr. Mercer contributes a paper on *The Teaching of Numerical Trigonometry.*

Two points suggested by that paper, I think, require further discussion. First, I hold that the pupils who are learning *Easy Numerical Trigonometry* are quite capable of appreciating a reasoned geometrical proof of the constancy of the ratios; and the experimental method advocated by Mr. Mercer, and others, seems to me, in the circumstances, wholly wrong. Secondly, the tendency to extend the work past the *Easy Numerical Trigonometry of the Right-Angled Triangle,* and to give it the dignity of a separate subject and a text-book of about 150 pages, instead of associating it with Elementary Geometry, Algebra and Arithmetic, is, I think, equally objectionable.

To take the first point: Mr. Mercer proceeds as follows: "... Let every boy draw an angle of 35°, AOB, take any point P in one arm and draw PM perpendicular to the other. Then let him measure carefully OM, MP and work out $\frac{MP}{OM}$ correct to two decimal places. They will all have different lengths for OM and MP, but should obtain the same value for $\frac{MP}{OM}$. This number (·70) is called the tangent of 35°, and a definition is suggested for the tangent of any acute angle. From any point P in one arm of the angle AOB, draw PM perpendicular to the other arm; then $\frac{MP}{OM}$ is called the tangent of the angle. ..."

If it were impossible, early in the school course, to give a reasoned proof that the ratio $\frac{MP}{OM}$ is independent of the position of the point P upon the arm of the angle, then we might, though unwillingly, put up with Mr.

Mercer's argument. But surely, when it is quite a simple matter to treat this question geometrically towards the beginning of the second year's work in Geometry, if not earlier, it is a pity to fall back upon one of these so-called proofs, with which we are becoming too familiar in the mathematical work of the school.

In dealing with parallels, the pupils have learned that if P_1, P_2, P_3, ... are any points at equal distances along one of two parallel lines (Fig. 1), and P_1p_1, P_2p_2, P_3p_3, etc., the perpendiculars from P_1, P_2, P_3, ... to the other line, then $P_1P_2 = p_1p_2 = p_2p_3$, etc.

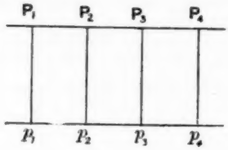


Fig. 1.

It is natural to examine what is the corresponding result in the case of any two intersecting lines OA and OB, enclosing an acute angle AOB. In this case it is equally easy to show that if P_1, P_2, P_3, ... are any points on one arm OB, such that $OP_1 = P_1P_2 = P_2P_3$, etc., and P_1p_1, P_2p_2, P_3p_3, etc., the perpendiculars from P_1, P_2, P_3, ... to the other arm (Fig. 2), then $Op_1 = p_1p_2 = p_2p_3$, etc.

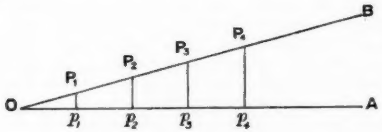


Fig. 2.

This theorem could, of course, be stated for any parallel lines cutting off equal intercepts from one of the intersecting lines.

Now let P and Q be *any* two points on the arm OB, and PM, QN the perpendiculars from P and Q to the other arm OA. We may assume that the segments OP and OQ are commensurable. Thus we may take $OP = m$ units, say equal to $m \,.\, OP_1$; and $OQ = n$ units, say equal to $n \,.\, OP_1$ (Fig. 3).

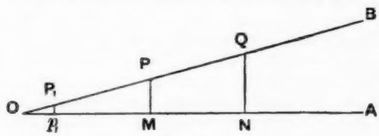


Fig. 3.

Then, from the theorem quoted above, we have

$$OM = m \,.\, Op_1 \text{ and } ON = n \,.\, Op_1,\ P_1p_1 \text{ being perpendicular to } OA.$$

Therefore $$\frac{OM}{OP} = \frac{ON}{OQ}.$$

Having established this result, it is a simple matter to treat the trigonometrical ratios—the sine, cosine, and tangent would alone be required in his numerical work. It is unnecessary to take the tangent first. Indeed,

from this point of view, it is more satisfactory to begin with the sine and cosine.

The ratios will be defined in the usual way: *Let OAB be any acute angle, and P any point upon one of the arms of the angle, say OB. From P draw the perpendicular PM to the other arm OA. Then the ratios*

$$\frac{MP}{OP},\ \frac{OM}{OP},\ \textit{and}\ \frac{MP}{OM}$$

are called the sine, cosine, *and* tangent *of the angle.*

The first thing to do is to prove that these ratios are independent of the position of the point P upon OB.

In the case of the sine, it is only necessary to draw the line through O perpendicular to OA, and from P the perpendicular PN to this line. Then we have $MP=ON$. But, by the geometrical theorem, $\frac{ON}{OP}$ is independent of the position of P upon OB. It follows that $\frac{MP}{OP}$ is also independent of the position of P upon OB.

The corresponding result for the ratio $\frac{OM}{OP}$ follows immediately from the geometrical theorem; and since

$$\frac{MP}{OM}=\frac{MP}{OP}\Big/\frac{OM}{OP},$$

the ratio $\frac{MP}{OM}$ is also independent of the position of the point P upon OB.

It is not difficult to go further, and show that, as the angle increases from $0°$ to $90°$, the sine and tangent continually increase, and the cosine continually diminishes. So that every acute angle has a definite sine, cosine, and tangent, which belongs to it and no other acute angle. If the angle is given, the ratios are known. If one of the ratios is given—a number between suitable limits—the angle is known.

In this argument there is nothing beyond the powers of the pupils early in their geometrical course. The experimental method may help in driving the point home; but the pupils themselves would be the first to admit that a proof was still lacking. Naturally the teacher will treat the different ratios separately, and illustrate each by numerous examples, and a certain amount of practical work. But, in my opinion, the geometrical theorem should be the foundation of the whole theory.*

Space does not permit me to do more than touch upon the second point referred to in my opening sentences. We are all agreed that it is advisable to introduce the *Easy Numerical Trigonometry of the Right-Angled Triangle* into the Secondary School Course quite early. But I submit that this can best be done in conjunction with the elementary work in the other mathematical subjects, Arithmetic, Algebra, and Geometry; and that it is unnecessary—indeed inadvisable—to extend this treatment further at this stage. The work can be made extremely interesting and instructive; and, if I may say so, Mr. Mercer's remarks upon the typical examples and illustrations are admirable. But where I disagree with him again—and I refer here to his text-book entitled *Numerical Trigonometry*, for the second part of his paper will not be in my hands yet for some time—is that I do not think it is a good thing at that early stage to cover in this way so large a part of the subject. Plane Trigonometry, as now generally taught, offers

* Cf. *A School Course in Geometry*, by W. J. Dobbs (Longmans' Modern Mathematical Series), Chapter IV. Mr. Nunn's *Teaching of Elementary Algebra* (*including the Elements of Trigonometry*), in the same series, has not yet reached me, but I shall be surprised if he does not adopt some such method as I have advocated above. The companion volume, *Exercises in Algebra* (*including Trigonometry*), is a most useful book.

little difficulty to the pupils of 15 years and upwards who learn it. No teacher will now fail to give prominence to the numerical work, even at the higher stage. But this work must be, and ought to be, different from that which many of us understand by the *Easy Numerical Trigonometry of the Right-Angled Triangle.* H. S. CARSLAW.

Sydney, Feb. 17, 1914.

430. [K[1]. 1.] The construction (and its justification) to solve the problem,

> "Given a finite straight line, it is required to cut off therefrom a segment equal to one-seventh of its length,"

may be made to depend entirely upon the first book of Euclid.

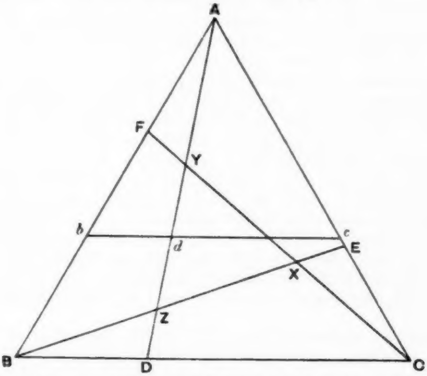
Describe any equilateral triangle Abc, and along bc set off $bd=(1/3)bc$. Produce Ad to D so that AD is equal in length to the given line; and through D draw BDC parallel to bdc, meeting AB, AC produced in B, C respectively. Along CA take $CE=BD$, and let BE intersect AD in Z. Then $ZD=(1/7)\ AD$.

Along AB take $AF=BD$ or CE, and let CF intersect BE, AD in X, Y respectively.

[Since $2bd=dc$, $2\triangle Abd=\triangle Adc$ and $2\triangle Bbd=\triangle Cdc$, wherefore

$$2\triangle ABd=\triangle AdC \text{ and } 2BD=DC.]$$

Contemplate the three-fold symmetry of the figure.



Since $AB=3AF$, $\triangle ABY=3\triangle AFY=3\triangle BDZ$ (of equal sides); hence $AY=3ZD$. Since $DC=2BD$, $\triangle AZC=2\triangle AZB=2\triangle CYA$ (of equal sides); hence $AZ=2AY$.

Thus $$AY=YZ=3ZD \quad\text{and}\quad ZD=(1/7)AD.$$

Generally, if $BD:DC=m:n$, $ZD=\{m^2/(m^2+mn+n^2)\}AD$, so that, for instance, if an equilateral triangle was described on AD and distances each $=ZD$ set off along the sides, we could obtain a segment $=1/43$rd of a certain line. R. F. DAVIS.

431. [B. 8. a.] *Reduction of a ternary cubic to the form*

$$X^3+Y^3+Z^3+6mXYZ$$

by a real transformation.

Prof. Elliott has given a proof of the reduction of a ternary cubic to canonical form in § 229 of his *Algebra of Quantics,* which (in the revised form which recently appeared in the second edition of this book) leaves

little to be desired from the point of view of simplicity. He does not prove however, that the reduction is possible by means of a real transformation, if the original cubic is real.

Weber, in his *Algebra*, vol. ii. § 107, gives a proof that the transformation is real, but his proof lacks the simplicity of Prof. Elliott's. The following proof may be interesting from the geometrical side.

The problem is equivalent to showing that a real cubic curve can be put in the form $X^3+Y^3+Z^3+6mXYZ=0$ by a real choice of homogeneous coordinates and triangle of reference.

Assume that the cubic has three real collinear inflexions. Take the tangents at these inflexions as the sides of the triangle of reference, and the line of inflexions as $x+y+z=0$. Then the cubic takes the form

$$(x+y+z)^3+6kxyz=0.$$

Now put $\quad x=aX+Y+Z,\quad y=X+aY+Z,\quad z=X+Y+aZ,$

where $\quad a=\{(2k)^{\frac{1}{3}}+2(9+2k)^{\frac{1}{3}}\}\div\{(2k)^{\frac{1}{3}}-(9+2k)^{\frac{1}{3}}\}$

is the real value of a satisfying

$$(a+2)^3+2k(a^2+a+1)=0,$$

and we find that

$$(a^2+a+1)\{(x+y+z)^3+6kxyz\}=(a-1)^2(X^3+Y^3+Z^3-3aXYZ).$$

We have assumed (i) that the cubic has three real collinear inflexions, (ii) that $a\neq 1$ and $\neq -2$, (iii) that the inflexional tangents are not concurrent.

We may justify (i) by projecting the cubic so that it has *one* real asymptote with ends A, B. The branch asymptotic at A must cross the asymptote at H (say) and be also asymptotic at B. It is at once evident from a diagram that, if the curve has no cusp or crunode (node with real tangents), there is a real inflexion between A and H, and another between H and B. The line joining these two real inflexions passes through a third real inflexion. (This method is, I believe, due to Chasles.)

With regard to (ii) and (iii), $a=1$ gives $k=-\frac{9}{2}$ and $(x+y+z^3)+6kxyz=0$ has an acnode (isolated point) at $(1, 1, 1)$; while $a=-2$ makes the inflexional tangents concurrent, so that $S=0$ and $T<0$. (Elliott's *Quantics*, §§ 292, 293.) The reader will readily supply the necessary proof in this case.

The form $(x+y+z)^3+6kxyz$ might also serve as a canonical form of the real ternary cubic, including the case in which the corresponding cubic curve has an acnode, but excluding the cuspidal and crunodal cubics and the case $S=0$, $T<0$.

From this form we at once deduce Clifford's theorem that such cubics may be projected so as to have a centre, and three-fold symmetry about the centre.

HAROLD HILTON.

432. [**6. a.** β.] *A Note on Momentum and Kinetic Energy.*

Many students find great difficulty in realising the fact that Momentum is a vector and Energy a scalar quantity.

The following treatment, starting from Newton's Law, sometimes seems to clear the minds of my students.

From $P=\dfrac{Wf}{g}$ and $v^2-u^2=2fs$, where all the vector quantities involved are in the same line, we easily establish the Energy equation

$$Ps=\frac{W(v^2-u^2)}{2g}.$$

Similarly $\quad Pt=\dfrac{W(v-u)}{g}.$

Now take a more general state of affairs, in which a body (W) is being

moved by forces P_1, P_2, P_3..., constant in direction and making angles θ_1, θ_2... with a fixed line.

As the body moves from position A to position B, where $AB=s$ and the direction of AB is ϕ, let the velocity change from u in a direction α at A to v in a direction β at B.

Resolving parallel to the fixed direction, it is easy to establish

$$(P_1\cos\theta_1+P_2\cos\theta_2+\cdots)s\cos\phi=\frac{W}{2g}(v^2\cos^2\beta-u^2\cos^2\alpha), \quad\text{.........(1)}$$

as well as

$$(P_1\cos\theta_1+P_2\cos\theta_2+\cdots)t \qquad =\frac{W}{g}(v\cos\beta-u\cos\alpha). \quad\text{............(2)}$$

In a perpendicular direction we have also

$$(P_1\sin\theta_1+P_2\sin\theta_2+\cdots)s\sin\phi=\frac{W}{2g}(v^2\sin^2\beta-u^2\sin^2\alpha) \quad\text{.........(3)}$$

and $$(P_1\sin\theta_1+P_2\sin\theta_2+\cdots)t \qquad =\frac{W}{g}(v\sin\beta-u\sin\alpha). \quad\text{............(4)}$$

Adding (1) and (3),

$$P_1s\cos(\theta_1-\phi)+P_2s\cos(\theta_2-\phi)+\cdots=\frac{W}{2g}(v^2-u^2), \quad\text{..................(5)}$$

so that by a happy juxtaposition of the cosines and sines all trace of the original fixed direction has been obliterated.

But anyone can see that equations (2) and (4) can never be rid of some reference to the fixed direction.

This way of getting (5) also brings out the fact that the directions of P_1, P_2, P_3... may be very different from one another, a fact which is often very imperfectly realised. W. M. ROBERTS.

433. [K. 8. a.] Mr. W. Finlayson sends me a proof much simpler and neater than my laboured one on Note 407, page 152. The three points of intersection of opposite sides of the hexagon $DECABF$ lie on a straight line. G, H are two of these points. The third, viz. where CE meets BF, is obviously the in-centre I.

He makes an interesting addition. Let AI meet BC in R and LM in J. Then, since $L(AHCK)$ is Harmonic, $L(AIRJ)$ is also Harmonic. $\therefore$ J coincides with the ex-centre I_1. So also I_2 lies on MN, and I_3 on NG.

E. P. ROUSE.

434. [J. 2. a.] *The Definition of Probability.*

The text-books usually define the probability of a favourable event by saying that if m events are deemed favourable and n unfavourable and all events are equally likely, the probability of a favourable event is $\frac{m}{m+n}$.

The rotatory character of this definition is manifest, and has been commented upon by many writers.

The theory made an advance, which is especially associated with the name of Dr. Venn, by explicit reference to the "long run" as part of the definition. Events are equally likely when they occur equally often in the long run.

A recent note* by M. Peano does not entirely remove the difficulty.

M. Peano says: "If a and b are classes, and the number of individuals in the class a is finite, the symbol $P(b, a)$ shall denote the number of a's which are b divided by the number of a's."

* *Atti della Reale Accademia dei Lincei,* vol. xxi. p. 429.

"If a is the class of possible cases, supposed finite, and b the class of favourable cases, the symbol $P(b, a)$, which is read Probability of a being b, means the ratio of the number of possible cases which are favourable to the total number of cases."

Here the element of chance seems to have disappeared, and we are left with a mere question of enumeration.

The following version of the definition is submitted.

Consider a set of a objects each A and of b each B.

Let the set be arranged in every possible sequence.

The number of possible sequences is $\dfrac{(a+b)!}{a!\,b!}=k$, say.

The number of these in which the rth place is occupied by A is

$$\frac{(a+b-1)!}{(a-1)!\,b!}=k_1 \quad\text{and}\quad \frac{k_1}{k}=\frac{a}{a+b}.$$

Now consider a number, very large compared with a or b, say $\mu k+\nu$.

Taking μ complete sets of possible sequences and ν others, the number which have A in the rth place lies between

$$\frac{\mu a}{a+b}k \quad\text{and}\quad \frac{\mu a}{a+b}k+\nu.$$

The ratio of the number of times the rth object in a set is A to the number of times it is B lies between

$$\frac{\dfrac{\mu a}{a+b}k}{\dfrac{\mu b}{a+b}k+\nu} \quad\text{and}\quad \frac{\dfrac{\mu a}{a+b}k+\nu}{\dfrac{\mu b}{a+b}k},$$

and this ratio therefore tends to the limit $\dfrac{a}{b}$, independently of the values of μ and ν, and the proportion of the objects in the rth place which are A tends to the value $\dfrac{a}{a+b}$.

If selections are made of sets of the A's and B's on some other principle, the selection is DEFINED as being fair if the ratio of the number of times A occupies the rth place to the number of trials tends to the limit $\dfrac{a}{a+b}$ when the number of trials is increased indefinitely.

C. S. J.

435. [M. 5.] *Surfaces generated by the Motion of an Invariable Cubic Curve.*

I have pointed out in these pages* that, if (ξ, η, ζ) is any point on a curve referred to moving axes rigidly attached to the curve, then each point P of the curve is moving at right angles to the tangent at P, provided $A\xi'+B\eta'+C\zeta'\equiv 0$; while the curve is always an asymptotic line on the surface it generates, if $\begin{vmatrix} A & B & C \\ \xi'' & \eta'' & \zeta'' \\ \xi' & \eta' & \zeta' \end{vmatrix}\equiv 0.$

In these formulae, $A=v_1-\omega_3\eta+\omega_2\zeta$, etc., where v_1, v_2, v_3 are the velocities of the moving axes parallel to their instantaneous positions, and ω_1, ω_2, ω_3 are their angular velocities about these instantaneous positions; while dashes denote differentiation with respect to a parameter u, of which ξ, η, ζ are functions.

I have applied these results to the most general twisted cubic curve, which may be thrown into the form

$$\xi=\frac{au^2+2bu+c}{u^3+3Hu+G},\quad \eta=\frac{pu+q}{u^3+3Hu+G},\quad \zeta=\frac{k}{u^3+3Hu+G},$$

* *Mathematical Gazette*, vii. (1913), p. 36.

by taking as axes of reference the tangent, principal normal, and binormal at the point $u=\infty$. The application of the formulae is quite straightforward, though the arithmetic is very tedious.

I find that the condition $A\xi'+B\eta'+C\zeta'\equiv 0$ is satisfied for all values of u if

$$v_1=v_2=0,\quad 3kv_3+ap\omega_3=0,\quad k\omega_2=q\omega_3,\quad k\omega_1=(c+aH)\omega_3.$$

The condition $\begin{vmatrix} A & B & C \\ \xi'' & \eta'' & \zeta'' \\ \xi' & \eta' & \zeta' \end{vmatrix}\equiv 0$ cannot be satisfied in general by any values of v_1, v_2, v_3, ω_1, ω_2, ω_3.

Hence we have the interesting result:

The most general twisted cubic curve can move so that each point P of the curve is always moving perpendicularly to the tangent at P, provided the cubic curve is screwed about a certain fixed axis (i.e. *traces out a certain helicoid*).

Again:

The most general twisted cubic curve cannot move so as to be always an asymptotic line on the surface it generates. HAROLD HILTON.

REVIEWS.

Das Problem der Kreisteilung. Von Dr. ARTHUR MITZSCHERLING; mit einem Vorwort von Dr. HEINRICH LIEBMANN. F. 210; S. 214. 1914. 8·40 m. (Teubner.)

The whole problem of the division of the circle has been considered here almost entirely from a geometrical standpoint. There are numerous figures given to illustrate the text and also several diagrams with the necessary explanations of mechanical devices for drawing certain well-known curves arising in the treatment of the subject.

Many pages are devoted to historical matter, but not to the entire exclusion of clear demonstrations of the various properties of the curves, although in some cases the historical and theoretical portions are not kept sufficiently separate. The earlier portion of the work is confined to constructing inscribed polygons with the use of the straight line and circle only. All of the better known approximations using only such constructions are given, in addition to a description of the mechanical circle-division method due to Reichenbach. The rest of the work treats mostly of the properties of curves that can be used for the trisection of an angle, and the remaining portion treats of the more general problems involved in polysection. A rigid proof of the impossibility, in general, of the trisection of an angle by means of the straight line and circle only, is followed by the consideration of various curves that can be used for trisection and polysection. Curves of higher order than the second, such as the trisectrix, cubical parabola, cycloid, etc., with more general curves, including that due to Ockinghaus of the *n*th order are dealt with, as well as the more elementary cases. Several approximate constructions are again given with a description of a few mechanical instruments.

The whole work has been carefully compiled, and in all cases references to the original papers are given in foot-notes, while an excellent index has been added at the end. Although the work is from its nature a collection of disconnected material, yet it has been brought together in such a way with careful and copious illustrations, as to make it quite an interesting book both for study and reference.

Leçons sur la Théorie des Nombres. Proféssées au College de France par A. CHÂTELET. Pp. 156. 1913. (Gauthier-Villars.)

This collection of lectures was delivered as an additional course to students already familiar with the Algebra of Serret and the French works on theory of numbers written by Tannery and Cahen. They were intended to serve as an introduction to some of the more recent researches due to Hermite and Minkowski.

In the first chapter some elementary theorems on forms and linear substitutions

are considered, together with the equivalence of general quadratic forms. The results are interpreted geometrically and Minkowski's generalisation of length and volume to n dimensions is introduced. The next four chapters deal with an explanation of the theory of moduli and their applications. A great deal of attention is paid to algebraic numbers and several arithmetical results of Hermite's relating to the transformation of Abelian functions are included in these chapters. This writer's well-known method of continuous reduction and its applications is found in the next chapter with the enunciations and proofs of two theorems given by Minkowski in his *Geometry of Numbers*. These theorems complete an extension of Hermite's work. The last chapter is devoted to the reduction of a base for an algebraic realm. Three notes are added at the end of the work on (1) the periods of functions, (2) an example of an algebraic realm, and (3) congruences with respect to an ideal and its norm.

It is interesting to find that some further effort is being made to collect and explain some of the more brilliant modern researches in theory of numbers. In this somewhat discontinuous series of lectures or essays an excellent attempt has been made to put some of Minkowski's work into a more accessible form for the ordinary student. At the same time its connection with the work of other authors is not forgotten, and the proofs of the various theorems are clear, following in the main the trend of the originals. Throughout, the geometrical standpoint of the subject is prominently brought out, and theorems and definitions clearly defined and emphasized. However, as in most French technical works, an index would form a valuable addition to the short table of contents. J. L. NAYLER.

A School Statics. By G. W. BREWSTER and C. J. L. WAGSTAFF. Pp. viii+248. 1914. 3s. net. (Heffer, Cambridge.)

This book, like most modern text-books on mechanics, adopts the historical and experimental development of the subject.

Work, Horse-power, Efficiency, etc., appear very early—in fact, in the first chapter; Levers and Moments come shortly after. The Parallelogram of Forces is rightly postponed till nearly halfway through the book, but in a chapter at the end formal bookwork is added by which the Parallelogram may be made to serve as a fundamental (experimental) proposition, and results deduced from it which in the earlier part of the text are based on the Principle of Moments. The order of the subjects may therefore easily be changed by individual teachers who desire to follow a different development of the subject.

Besides the experimental proofs of the fundamental proposition a number of other experiments are suggested for the pupil to do himself.

As to the scope of the book, it covers all the ground of elementary two-dimensional Statics and introduces its reader to Bow's Notation, Hooke's Law, Virtual Work, Bending Moments, and even Indicator Diagrams.

The most striking thing about the book is the large number of examples (over 700), many of which are extremely interesting, referring as they do to matters of every-day life—bicycles, motor-cars, turbines, lathes, etc., etc.

The figures are bold and well printed, and the "get-up" of the book attractive.

The authors are to be congratulated on the production of a book which must be interesting to the student who uses it, as they keep before him the fact that Statics is a subject of living interest and not a collection of abstract propositions.

If a review is only to be considered complete if it finds fault, it may be pointed out that though the word "force" is used freely from the very beginning, no explanation of the idea of force with its necessary accompaniment of action and reaction occurs till the fourth chapter. The interesting first few pages of that chapter might well come sooner.

Plane and Spherical Trigonometry. By ROBERT E. MORITZ, Ph.D., Ph.N.D. Pp. xvi+357+96. 10s. 6d. net. 1913. (Chapman, Hall.)

The present treatise consists of two parts, the first dealing with plane and the second with spherical Trigonometry. The section on plane trigonometry deals with the functions of an acute angle in the order indicated under the following headings: solution of right-angled triangles; logarithms; logarithmic solution of right-angled triangles; functions of an obtuse angle; properties of triangles; solution of oblique triangles; the general angle and its measures; functions of two or more angles; trigonometric equations and curves; complex numbers; trigono-

metric series and calculations of tables; hyperbolic functions. The section on spherical trigonometry treats in the order indicated of right and quadrantal spherical triangles; properties of oblique spherical triangles; solution of oblique spherical triangles.

The book aims throughout at presenting the subject from the modern point of view. For this reason there is abundance of numerical work leading up to the various generalisations, and the examples are drawn from practical life to a large extent and from geography and navigation. Much of the work dealing with the numerical calculations involved in constructing tables is of great value to the student. Rough checks and the calculation of the errors involved in using the various series are given throughout. The historical notes scattered through the text will give the student an excellent idea of how the subject has grown.

On the other hand, the book is too big. The mere bulk of the book and the sight of such a vast amount of matter to be covered would discourage many students at the very outset. The book could have been reduced to half or two-thirds its present size without any detriment to its usefulness by omitting much of the detail.

The treatment of Limits and Series proceeds on the time-honoured archaic lines. Many teachers will regret that no preliminary discussion on the Theory of Number has been given, so that the treatment of Series may be allowed to rest on its modern basis.

Analytical Geometry and Principles of Algebra. By Alexander Ziwet and Louis Allen Hopkins. Pp. viii + 369. 1.60$. 1914. (Macmillan Co.)

The above work (published in America) attempts to deal in limited space with the principles of Algebra in so far as they bear upon Plane and Solid Geometry. The subjects treated are: (1) Coordinates; (2) The Straight Line; (3) Simultaneous Linear Equations—Determinants; (4) Relations between Two or more Lines; (5) Permutations and Combinations—Determinants of any order; (6) The Circle—Quadratic Equations; (7) Complex Numbers; (8) Polynomials—Numerical Equations; (9) The Parabola; (10) Ellipse and Hyperbola; (11) Conic Sections—Equations of Second Degree; (12) Higher Plane Curves; (13) Coordinates (in three Dimensions); (14) The Plane and the Straight Line; (15) The Sphere; (16) Quadric Surfaces—Other Surfaces. It will thus be seen that the scope of the book is quite elementary, and that a close coordination between Algebraical Theorems and their Geometrical Interpretations is kept in view throughout.

There are many good things in the book. From one point of view m in $y=mx+c$ is regarded as the rate of increase of y with respect to x. Determinants are introduced as useful functions required immediately for geometrical work. The fundamental theorem with regard to the derangements of the suffixes in $a_1, b_2, c_3 \ldots l_n$ is not taken for granted without apology (as in most text-books), but a thoroughly clear and lucid proof is given. The introduction to Irrational Numbers and Complex Numbers is excellent and noteworthy as a piece of modern pedagogy. No attempt is made to enter into the philosophic basis of these extensions of the idea of "number," but the introduction of these new kinds of number is shown to be "historically determined very largely by the *applications* of arithmetic and algebra." There is an excellent set of examples on Argand's Diagram.

On the other hand, one feels that though every man can get a bite, yet no man can get a full meal. The modern tendency to "correlate" all the branches of a subject without overburdening with details has been carried to excess in this book. Thus one can learn practically nothing from the pages on "Higher Plane Curves." This subject is so vast that a few pages devoted to it only allow the explorer to jump the fence without penetrating the forest. It is a pity to find the Equation of the Tangent to a Circle by a mere dodge. Joachimsthal's Method or a treatment by the Method of Limits would have taught the student more.

The general impression left by the book is that it is not practical enough for engineers, and not formal enough for the academic student. On the other hand, it must be said that it is difficult for us on this side of the water to pass a just criticism on a text-book intended primarily for use in America.

William P. Milne.

New Tables of Logarithms. By E. MOUGIN. Pp. 56. 1s. 3d. Published by the Author. (Lycée, Roanne, France.)

At a time when the practical use of logarithms has to struggle against the combined use of tables of natural functions and calculating machines, even small improvements in the compactness and facility of a table are valuable. Of recent years Professor Huntington of Harvard University has produced what must be near finality in four-figure tables: and now M. Mougin has succeeded in producing a five-figure table in large bold type in nine pages, of about $6\frac{1}{2}$ inches each way.

The arrangement is new and ingenious. Some idea may be obtained by the line shown, in which logarithms and parts of logarithms are in heavier type:

								5			
								0	**0**	**1**	**2**
	90							$\underline{\mathbf{4}}$	**9**	$\underline{\mathbf{5}}$	**0**
80		0	1	2	3	4	5	6	7	8	9
		3									

Whence, for instance, $\log 8{\cdot}035 = {\cdot}90504$, $\log 8{\cdot}039 = {\cdot}90520$.

[Forced figures are underlined by M. Mougin.]

Dr. Glaisher once said that the only way to test a new table for convenience was to compute with it steadily for some hours on end. We believe that tried by this test most four-figure tables would come out very badly, even in comparison with seven-figure tables. This, the computer's point of view, is of course not the same as the teacher's: but it is perhaps worth some notice.

Time must show how far M. Mougin's arrangement satisfies the requirement of not proving tiring to the computer; but it is to be hoped that the labour and ingenuity expended will be appreciated. Smaller portions of the tables (omitting or shortening the tables of trigonometrical functions) are obtainable at 5d. and 10d. respectively.

Problèmes de Mécanique et Cours de Cinématique. Par C. GUICHARD. Redaction de MM. Dautry et Deschamps. Pp. 156. 6 fr. 1913. (A. Hermann, Paris.)

This book is based upon lectures by Professor Guichard to students of Mechanics. The first 83 pages contain some very beautiful applications of elementary calculus to various mechanical problems. For instance, at p. 49, there is an elegant investigation of the position of the centre of gravity of a catenary. These problems are followed by a chapter on Kinematics, which completes the book. C. S. J.

Alle Jahreskalendar auf einem Blatt. Von Dr. DOLIARIUS (J. E. Böttcher). Price 30 pf. 1914. (Teubner, Leipzic.)

An ingenious arrangement, by which a card, of the size of a reply postcard, and one moveable slip can be adjusted so as to give the calendar of any year. On the left of the opened card is a table of Easters.

By means of the date of Easter Sunday an opening in the moveable slip is adjusted over the right-hand half of the card, and the calendar is ready.

QUERIES.

(89) On page 122 of his *Elementary Treatise on Cross-Ratio Geometry*, the Rev. J. J. Milne has the following footnote:

"On the outside cover of an Appendix (1847) to Potts' larger edition of Euclid there was a notice that it was proposed to publish by subscription a translation of Simson's *Restoration of the Porisms*. The translation was to be preceded by a discussion of their peculiar character, together with a full development of the algebraical method of investigating them."

"If a number of subscribers had been obtained sufficient to defray expenses, it was intended to print the work at the University Press, in octavo, and to issue it at a price not exceeding ten shillings."

In the *Mathematician* for July 1849 (iii. 312), T. S. Davies writes: "... I am not without the hope that Mr. Potts' translation of the [Porisms] ... with valuable explanatory notes and illustrations, will not be long delayed." And in the *Mathematician* for September 1850 (iii. supplementary number, page 42), occurs this sentence from the pen of T. S. Davies: "In the notes on Mr. Potts' translation of Simson's Porisms, I shall give a sufficiently full account of Mr. Noble's views. . . ."

Was Potts' translation ever published?

(90) I have references to two works by "T. Leybourn," who is, presumably, the Thomas Leybourn (1770-1840) who was teacher of Mathematics at the Royal Military College, Sandhurst, and editor of the two series of the *Mathematical Repository* (1795-1835). The works are: 1. "System of Theoretical Geometry, 1813." 2. "Geometrical Solutions, London, 1818." Can anyone describe these works, or tell me where copies may be seen? A copy of the latter is particularly desired.

Neither of these works is mentioned in the *Dictionary of National Biography* or the *British Museum Catalogue*.

(91) Before me lies an incomplete work with the following title page: A | Treatise | concerning | Porisms | By Robert Simson, M.D. | In which the Author hopes that the Doctrine | of Porisms is sufficiently explained and for the future | will be safe from Oblivion. | Translated from the Latin | By John Lawson, B.D. | Canterbury, | Printed and Sold by Simmons and Kirkby; | Sold also by J. Nourse, B. White, J. Robson, Booksellers in | London, Merrils at Cambridge and Prince at Oxford. | MDCCLXXVII.

The whole contains 40 pages (vi + 34) and "Plate I." with "XVIII." figures. On page 34 only the first part of Proposition XVII. is given, and reference is made to a "figure XX." The body text of the page ends abruptly in the middle of a sentence: "But there is another rectangle $HG.FE$; therefore $HE.GF : HG.FE :: EF.HM : HG.FE :: HM :$"

Simson's work, "De Porismatibus Tractatus; quo doctrinam porismatum satis explicatam, et in posterum ab oblivione tutam fore sperat auctor," occupies pages 315-594 of his *Opera Quaedam Reliqua* (Glasguae, M.DCC.LXXVI) and contains "XCIII" propositions. The above-mentioned fragment by Lawson is a translation of pages 315-380. Did he publish a further translation? His biographer in the *Dictionary of National Biography* gives in a list of his works (the italics are mine): 4. "A Treatise concerning *Prisms* by Robert Simson, M.D., translated from the Latin, 4to, Canterbury, 1777." There is no copy of the work in the British Museum.

In the advertisement at the end of the 1821 edition of Thomas Simpson's *Elements of Geometry*, I find listed: "Simson's (R.), Treatise on Porisms, by Lawson, 4to, 3s. 6d." and "Lawson's (Rev. J., F.R.S.) Mathematical Works—containing . . . A Translation of Dr. R. Simson's Treatise on Porisms . . . in one volume, 4to, 21s." Nevertheless this same list gives Lawson's "Synopsis of all the Data for the Construction of Triangles,* 4to, 2s. 6d." And as this pamphlet contains only 24 pages, a 40 page pamphlet might well cost "3s. 6d."

In the *Mathematician* for July 1849 (iii. 313), T. S. Davies (a most careful and accurate writer) remarks: ". . . it is less to be regretted that Lawson did not complete his translation than it otherwise might have been."

But did Lawson publish anything beyond the fragment described above?

Brown University, Providence,
Rhode Island, U.S.
R. C. ARCHIBALD.

* The first edition of 16 pp. was published in 1723 at Rochester, price 1s. (W. J. G.).

THE MATHEMATICAL ASSOCIATION.

At a Special General Meeting held at the London Day Training College, Southampton Row, W.C., on Saturday, 16th May, 1914 (Mr. Alfred Lodge in the Chair), it was resolved :

(i) That the existing Rule relating to "Local Branches and Associates" be rescinded and that the following new Rule be substituted :

VI. Local Branches and Associates.

Local Branches of the Association may be formed, or existing Societies recognised as Local Branches of the Association, with the approval of the Council.

All Local Branches shall have power to elect Associates of the Association.

An Associate shall be entitled to attend all meetings of the Association, to use the Library, and to receive a copy of the List of Members, the Rules, and the Reports issued on behalf of the Association. An Associate shall not be entitled to receive the *Mathematical Gazette* or to vote on any matter of business.

Each Branch shall pay to the Association an annual subscription of ten shillings, and one copy of the *Mathematical Gazette*, as issued, shall be supplied to each Branch.

Each Branch shall determine for itself the annual subscription to be paid to that Branch by each Associate belonging to it.

The Secretary of each Branch shall furnish annually to the Secretaries of the Association a list of the Members and Associates attached to that Branch, and a copy of the Annual Report or Record of proceedings of the Branch.

The Association shall pay to each Branch the sum of one shilling and sixpence annually on behalf of each Ordinary Member of the Association who is attached to that Branch. The Branch shall not be entitled to any further subscription from any such Member beyond the amount by which the annual subscription paid to the Branch by each Associate shall exceed the sum of one shilling and sixpence. In any year the payment of one shilling and sixpence shall not be made to more than one Branch in respect of any one Member of the Association, and the total annual payment to any one Branch under this rule shall not exceed Four pounds.

(ii) That the following clause be added to those defining the Constitution of the "Girls' Schools" Special Committee.

(*e*) The Committee shall have power to co-opt not more than three others, who need not necessarily be Members or Associates of the Association at the time of their election, but who shall have no vote until they have joined the Association either as Members or as Associates.

The Mathematical Association.

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Co-opted by the Committee.

MISS C. M. WATERS, County School for Girls, Bromley, Kent.

* Will serve for four years.

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THE Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue has been issued to members containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

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BOOKS, ETC., RECEIVED.

The American Journal of Mathematics. Edited by F. MORLEY. Vol. XXXVI. No. 2. April 1914. 5$ per ann. (The Johns Hopkins Press, Baltimore.)

Iterated Limits in General Analysis. R. E. ROOT. *Simple Transitive Primitive Groups whose Maximal Subgroup contains a Transitive Constituent of Order p^2 or pq, or a Transitive Constituent of Degree 5.* E. R. BENNETT. *An Extension of Green's Theorem.* IDA BARNEY. *On the Asymptotic Solutions of Linear Differential Equations.* C. E. LOVE. *Restricted Systems of Equations.* A. B. COBLE. *The Canonical Types of Modular Conics.* A. H. WILSON. *On Long Waves.* J. H. WEDDERBURN.

Outdoor and Indoor Experimental Arithmetics. Teacher's Book. By H. GOODACRE, E. F. HOLMES, C. F. NOBLE, PERCY STEER. 3s. 6d. net. 1914. Pp. xii + 376.

Die Berührungstransformationen Geschichte und Invariantentheorie. By H. LIEBMANN and F. Engel. Pp. 80. 3 m. 1914. (Teubner, Leipzic.)

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Elementare Algebra. Akademische Vorlesungen für Studierende der ersten Semester. By E. NETTO. 2nd edition. Pp. x+200. 4.40 m. 1913. (Teubner, Leipzic.)

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The Mathematics Teacher. Vol. VI. No. 3. March 1914. 1 $ per ann. (Association of Teachers of Mathematics for the Middle States and Maryland, Lancaster, Pa.)

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Test Papers in Elementary Algebra. By C. V. DURELL. Pp. viii+233. 3s. 6d. 1914. (Macmillan.)

The Algebra of Logic. By L. COUTURAT. Authorised Translation by L. G. Robinson, with Preface by P. E. B. Jourdain. Pp. xiv+98. 1.50$. 1914. (Open Court, Chicago.)

An Elementary Treatise on the Calculus, for Engineering Students. By J. GRAHAM. Pp. x+355. 4th edition. 5s. net. 1914. (Spon.)

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Abhandlungen über jene Grundsatze der Mechanik, die Integrale der Differentialgleichungen liefern von Isaac Newton (1687), *Daniel Bernoulli* (1745) *und* (1748), *und Patrick d'Arcy* (1747). Herausgegeben von P. E. B. JOURDAIN. Pp. 109. No. 191. Ostwald's Klassiker der exakten Wissenschaften. 2.80 m. 1914. (Engelmann, Leipzig.)

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Arithmetic. By W. M. BAKER and A. A. BOURNE. 2nd edition. Part I. Pp. xii+195+xxix. Part. II. Pp. 196-400+lii. Answers perforated. 2s. each. 1914. (Bell & Sons.)

Statics. Part I. By R. C. Fawdry. Pp. vi+165. 2s. 6d. 1914. (Bell & Sons.)

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Darstellende Geometrie. By J. HJELMSLEV. Pp. ix+320. 5.40 m. Geb. in Leinwand 6 m. 1914. (Teubner, Leipzic.)

Praktische Analysis. By H. von SANDEN. Pp. xix+185. 3.60 m. Geb. in Leinwand 4.20 m. 1914. (Teubner, Leipzic.)

The Theory of Relativity. By L. SILBERSTEIN. Pp. viii+295. 10s. net. 1914. (Macmillan.)

A First School Calculus. By R. WYKE BAYLISS. Pp. vii+288. 4s. 6d. 1914. (Arnold.)

Memorabilia Mathematica, or the Philomath's Quotation-Book. By R. E. MORITZ. Pp. x+410. 12s. 6d. net. 1914. (Macmillan Co.)

Time as a Fourth Dimension. By R. C. ARCHIBALD. Pp. 4. Reprinted from Bull. Amer. Math. Soc. 2nd Ser. Vol. XX. No. 8. 1914.

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